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ON SWITCHING PATHS POLYHEDRA

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A class of integer polyhedra with totally dual integral (tdi) systems is proposed, which generalizes and unifies the "Switching Paths Polyhedra" of Hoffman (introduced in his generalization of Max Flow-Min Cut) and such polyhedra as the convex hull of (the incidence vectors of) all "path-closed sets" of an acyclic digraph, or the convex hull of all sets partitionable into k path-closed sets. As an application, new min-max theorems concerning the mentioned sets are given. A general lemma on when a tdi system of inequalities is box tdi is also given and used.

1. Introduction

Consider an acyclic digraph $G=(V, E)$ with node set V and edge set E and let P be some path of G with node set $V(P)$. Call a 0, ± 1-vector $A_0 \in \mathbf{R}^V$ an *alternating vector of P in G* if its support is contained in $V(P)$, and traversing P and ignoring the zero-components, the $+1$ and -1 alternate in an sequence $+1, -1, +1, \ldots$ $\ldots, -1, +1$. Let A_i, $i \in I$, be all alternating vectors of all paths of G. Furthermore, call a node set $T \subseteq V$ *path-closed* if $v, v' \in T$ and $w \in V$ is on a path from v to v' implies $w \in T$ (this notion is close to that of a convex set of a poset and we refer the interested reader to [6]). One result of [6] is that the polyhedron

$$\{x \in \mathbf{R}^V \colon x \geqq 0,\ A_i x \leqq 1,\ i \in I\} \tag{1.1}$$

is the convex hull of all path-closed sets of G, and was proven by means of the anti-blocker to (1.1) and by algorithms.

Now, this result is essentially the statement that (1.1) is an integer polyhedron, and it was natural to look for a proof of this statement using the general and powerful tool of total dual integrality. In this task, an operation of *switching* of alternating vectors showed to be fruitful. This concept is similar to the switching of paths used in [9] by Hoffman in order to establish a class of integer polyhedra of the type $\{x \geqq 0 \colon Ax \geqq r\}$, where A is the incidence matrix of a family of abstract paths *closed with respect to switching*, and where r is in a certain sense *supermodular*.

This observation motivated the development of the present class of integer polyhedra, *called switching paths polyhedra.* This class unifies and generalizes the polyhedron (1.1) and the polyhedra described in [9] by Hoffman. Our approach is very close to his: in fact, it extracts the essence of his arguments to use them in a somewhat more general framework.

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In the next section, we introduce the switching paths polyhedra and state the main theorem. In section 3, we give some examples and applications and in section 4 prove two preparatory lemmata, one of which relating the properties of total dual integrality and box total dual integrality. Section 5 deals with the proof of the main theorem.

2. Switching families

Let U and A be finite sets and $\mathfrak{s}=\{f(a)\colon a\in A\}$ a family of 0, ±1-vectors $f(a)\in\mathbf{R}^U$, $a\in A$. Denote for any vector $f\in\mathbf{R}^U$ with components $f_j, j\in U$, by s^+f the *positive support* of f, ie. $s^+f=\{j\in U\colon f_j>0\}$, by s^-f the *negative support*, and by $sf=s^+f\cup s^-f$ the *support* of f. Let further be given for any $a\in A$ a *linear ordering* $\underset{a}{\leq}$ of $sf(a)$ and define for any $a, b\in A$ with $j\in sf(a)\cap sf(b)$ the following sets:

$$[aj]=\{i\in sf(a)\colon i\underset{a}{\leqq}j\},\quad [jb]=\{i\in sf(b)\colon i\underset{b}{\geqq}j\}\quad\text{and}\quad [ajb]=[aj]\cup[jb].$$

Definition 2.1. A family $\mathfrak{s}$ as above is called a *switching family* if for any $a, b\in A$ and $j\in sf(a)\cap sf(b)$ there exists c and $d\in A$, denoted *ajb* and *bja*, such that the following properties hold:

$$sf(ajb)\cap sf(a)\cap sf(b)\subseteqq[ajb], \tag{2.1}$$

$$\left.\begin{array}{ll} s^*f(ajb)\cap[bj]\subseteqq s^*f(a) & (2.2)\\ s^*f(ajb)\cap[ja]\subseteqq s^*f(b) & (2.3)\end{array}\right\}\quad\text{for}\quad {}^*=+,-,$$

similar properties for *bja*, to which we refer later as (2.1)′, (2.2)′ and (2.3)′, and

$$sf(ajb)\cap sf(bja)\subseteqq sf(a)\cap sf(b). \tag{2.4}$$

In order to facilitate the interpretation of these properties, we give some examples of switching families:

Example 2.2. In the digraph $G=(U, E)$ with source s and sink t, let $f(a)\in\mathbf{R}^E$, $a\in A$, be the edge set incidence vectors of all $s-t$-paths P_a of G. The $f(a)$'s being naturally ordered, for any $a, b\in A$ and edge $j\in P_a\cap P_b$, let $f(ajb)$ be the vector of a path P_{ajb} whose edges are contained in $\{i\in P_a\colon i\underset{a}{\leqq}j\}\cup\{i\in P_b\colon i\underset{b}{\geqq}j\}$. (The paths being identified here with their (ordered) edge sets.)

Example 2.3. In the acyclic digraph $G=(U, E)$, let $f(a)\in\mathbf{R}^U$, $a\in A$, be the node set incidence vectors of all paths P_a of G. The $f(a)$'s being again ordered in the obvious way, for any $a, b\in A$ and node $j\in P_a\cap P_b$, let $f(ajb)$ be the vector of the path

$$P_{ajb}=\{i\in P_a\colon i\underset{a}{<}j\}\cup\{j\}\cup\{i\in P_b\colon i\underset{b}{>}j\}, \tag{2.5}$$

(the paths being identified here with their node sets).

Example 2.4. In the previous graph, let $f(a)\in\mathbf{R}^U$, $a\in A$, be all alternating vectors of paths of G (see section 1 for the definition). Also, for any $a, b\in A$ and $j\in sf(a)\cap sf(b)$, if P_a and P_b are two paths of which $f(a)$ and $f(b)$ are alternating vectors, let P_{ajb} be given by (2.5) and $f(ajb)$ by

$$f_i(ajb) = \begin{cases} (f_i(a)+f_i(b))/2 & \text{for} \quad i=j, \\ f_i(a) \quad \text{for} \quad i\in P_a & \text{and} \quad i \underset{a}{<} j, \\ f_i(b) \quad \text{for} \quad i\in P_b & \text{and} \quad i \underset{b}{>} j, \\ 0 \quad \text{otherwise} \end{cases}$$

Observe that $f(ajb)$ is an alternating vector of P_{ajb}.

Definition 2.5. Given a family $\mathfrak{s}=\{f(a)\colon a\in A\}$, a function $h\colon A\to\mathbf{R}$ is said to be *submodular on A* if for any $a, b\in A$ and $j\in sf(a)\cap sf(b)$, $h(ajb)+h(bja)\leqq h(a)+h(b)$, *supermodular on A* if $-h$ is submodular on A, and *modular on A* if h is both super- and submodular on A.

Definition 2.6. Given a family $\mathfrak{s}=\{f(a)\colon a\in A\}$, let $F\to\mathbf{R}^{A\times U}$ be the matrix with rows $f(a)$, $a\in A$. The family $\mathfrak{s}$ is called *submodular (modular)* if all columns f_j, $j\in U$, of F are submodular (modular) functions on A.

Remark 2.7. The switching family of Example 2.2 is submodular and those of Examples 2.3 and 2.4 are modular.

We recall that a polyhedron P is said to be an *integer polyhedron* if each non empty face of P contains an integer point. In particular each vertex of P is integer. Also, given rational matrix Q and vector q, the system of linear inequalities $Qx\leqq q$ is said to be *totally dual integral* (tdi) if for any integer vector c such that min $\{qy\colon y\geqq 0, yQ=c\}$ exists, this minimum is achieved by an integer y. Finally, a well-known sufficient condition for $P=\{x\colon Qx\leqq q\}$ to be an integer polyhedron is that $Qx\leqq q$ is tdi and q is integer [3,9].

We can now state the main result.

Theorem 2.8. *Let $\mathfrak{s}=\{f(a)\colon a\in A\}$ be a switching family, $F\in\mathbf{R}^{A\times U}$ the matrix with rows $f(a)$, $a\in A$, $r\colon A\to\mathbf{Z}$ an integer valued function and $e, d\in\{\mathbf{Z}\cup\pm\infty\}^U$. Then the following polyhedra are integer polyhedra with tdi systems:*

a) $P_1=\{x\in\mathbf{R}^U_+\colon e\leqq x\leqq d,\ Fx\geqq r\}$ *for $\mathfrak{s}$ submodular and r supermodular on A.*

b) $P_2=\{x\in\mathbf{R}^U\colon e\leqq x\leqq d,\ Fx\leqq r\}$ *for $\mathfrak{s}$ modular and r submodular on A.*

3. Examples and applications

3.1. In [9], to generalize and unify the different versions of the max flow min cut theorem, Hoffman considers a family $\mathfrak{s}=\{S_0, S_1, \ldots, S_m\}$ of subsets $S_k\subseteqq U$, with $S_0=\emptyset$ and each $S_k\neq\emptyset$ being linearly ordered by a relation $\underset{k}{<}$. Furthermore, $\mathfrak{s}$ is "closed with respect to switching", i.e. for any $S_k, S_l\in\mathfrak{s}$ and $j\in S_k\cap S_l$, there exists $S_t\in\mathfrak{s}$, which we denote $S_{(kjl)}$, such that $S_{(kjl)}\subseteqq\{i\in S_k\colon i\underset{k}{<}j\}\cup\{j\}\cup\{i\in S_l\colon i\underset{l}{>}j\}$. Analogously, there is $S_{(ljk)}\in\mathfrak{s}$. Moreover, a function $r\colon\mathfrak{s}\to\mathbf{Z}_+$ is given, satisfying $r(\emptyset)=0$, and, for any S_k, $S_l\in\mathfrak{s}$ and $j\in S_k\cap S_l$, $r(S_{(kjl)})+r(S_{(ljk)})\geqq r(S_k)+$ $+r(S_l)$. Now the polyhedron $\{x\in\mathbf{R}^U_+\colon\sum\{x_j\colon j\in S_k\}\geqq r(S_k),\ k=1, \ldots, m\}$ of

Theorem 2.4 of [9] is of the type P_1 of Theorem 2.8: let $A=\{1, \ldots, m\}$, and for any $a \in A$, $f(a) \in \mathbf{R}^U$ be the incidence vector of S_a and the ordering of $sf(a)$ be the given order of S_a. The hypotheses for F and r of P_1 are then easily verified.

3.2. Let $G=(U, E)$ be an acyclic digraph, $f(a) \in \mathbf{R}^U$, $a \in A$, be the node set incidence vectors of all paths P_a of G (Example 2.2), and $r(a)=1$ for all $a \in A$. Then $\{x \in \mathbf{R}^U: f(a)x \leqq 1,\ a \in A\}$ is of type P_2 and is a polyhedron of [8], whose vertices are the 0, ± 1-vectors associated with the *node cut sets* introduced there. Similarly, $\{x \in \mathbf{R}^U:\ x \leqq 1, f(a)x \leqq k\}$ for k a positive integer, is of type P_2.

3.3. Let $G=(U, E)$ be the *comparability graph* of a poset U with ordering $\leqq$, i.e. for any $i, j \in U$, $(i, j) \in E$ iff $i<j$, and let $f(a)$, $a \in A$, be defined as in 3.2. The polyhedra of type P_2 $\{x \in \mathbf{R}^U_+: f(a)x \leqq 1,\ a \in A\}$ and $\{x \in \mathbf{R}^U_+:\ x \leqq 1,\ Ax \leqq k\}$, where k is a positive integer, are the convex hulls of the *antichains* and of the *sets partitionable into k antichains* of U. Moreover, substituting k by a modular function r in the second polyhedron, one obtains the chain model of Hoffman and Schwartz [10].

3.4. Let $G=(U, E)$ be an acyclic digraph, $f(a) \in \mathbf{R}^U$, $a \in A$, be all alternating vectors of paths of G (Example 2.4), and $r(a)=1$ for all $a \in A$. $\{x \in \mathbf{R}^U_+:\ f(a)x \leqq 1,\ a \in A\}$ is of the type P_2 and is the convex hull of the path-closed sets of G (see (1.1)).

3.5. Let G and $f(a)$, $a \in A$, be as in 3.4 and k be a positive integer. $\{x \in \mathbf{R}^U_+:\ x \leqq 1,\ f(a)x \leqq k,\ a \in A\}$ is of type P_2, and it follows from Corollary 3.4 of [6] that this polyhedron is the convex hull of the *sets partitionable into k path-closed sets.*

3.6. Since switching paths polyhedra have their systems tdi, they are suitable for deriving combinatorial min-max theorems. As such corollaries, we only mention for the polyhedra of 3.1 the min flow-max cut theorem [9], for 3.2., min-max theorems yielding, as Cameron shows in [1], in an elegant way the theorem of Gallai and Milgram [4] (and a generalization of it [1]) for acyclic digraphs, for 3.3, Dilworth's theorem [2] and its generalizations by Greene and Kleitman [5] and by Hoffman and Schwartz [10]. For 3.4 and 3.5, we give the following results.

Let U be a finite poset with ordering $\leqq$ and $G=(U, E)$ of 3.4 be its comparability graph. In the theory of ordered sets, a set $T \subseteqq U$ is called *convex* if $i, k \in T$ and $j \in U$ is such that $i \leqq j \leqq k$ implies $j \in T$. Therefore the convex sets of U are the path-closed sets of G.

Given any $S \subseteqq U$, say that a family of subsets of U *covers* S if any $j \in S$ is in some member of the family, and *is packed in* S if no $j \in S$ is in more than one member of the family. Also, given a partition U^+, U^- of U ($U^+ \cup U^- = U$ and $U^+ \cap U^- = \emptyset$), call a chain with ordered elements $j_1, j_2, \ldots, j_{2k+1}$ an *alternating chain with respect to (w.r.t.)* U^+, U^- if $j_i \in U^+$ for i odd, and $j_i \in U^-$ for i even. Finally, for any $S \subseteqq U$, call $|S \cap U^+| - |S \cap U^-|$ the *surplus of S w.r.t.* U^+, U^-. Applying the duality theorem satisfied with integer solutions to the lp "maximize cx s.t. $x \geqq 0$, $f(a)x \leqq 1$, $a \in A$", yields for any $c \in \mathbf{Z}^U$ a combinatorial min-max theorem. In particular, for $c_j = {}^*1$ for $j \in U^*$, ${}^* = +, -$, we have:

Theorem 3.1. [6] *Given a partition U^+, U^- of the poset U, the maximum surplus of a convex set of U w.r.t. U^+, U^-, is equal to the minimum number of alternating chains w.r.t. U^+, U^-, which cover U^+ and are packed in U^-.* ∎

This result is analogue to Dilworth's theorem and also has a polar counterpart. For any $S \subseteq U$ and alternating chain C w.r.t. $S, U-S$, call $|C \cap S|$ the *length in S of C.*

Theorem 3.2. [6] *Given $S \subseteq U$, the minimum number of convex sets partitioning S is equal to the maximal length in S of an alternating chain w.r.t. $S, U-S$.* ∎

In the same way as the theorem of Greene and Kleitman generalizes Dilworth's theorem, the following result generalizes theorem 3.1.

Theorem 3.3. *Given a partition U^+, U^- of the poset U and an integer $k \geqq 1$, the maximum surplus w.r.t. U^+, U^- of a set partitionable into k convex sets is equal to the smallest number expressable as $|S|+k\cdot\alpha(U-S)$, where $S \subseteq U^+$ and $\alpha(U-S)$ is the minimum number of alternating chains w.r.t. U^+-S, U^-, which cover U^+-S and are packed in U^-.*

Proof. Consider the polyhedron of 3.5, where G is the comparability graph of U. Since its system is tdi, for any $c \in \mathbf{Z}^U$, the lp "minimize $ky+1u$, $y \in \mathbf{R}_+^A$, $u \in \mathbf{R}_+^U$, s.t. $yf_j+u_j \geqq c_j$, $j \in U$", has an integer optimal solution y^*, u^*. To prove the theorem, we must show that we can assume additional properties for y^*, u^* besides integrality.

First, we shall see in the proof of theorem 2.8 that we may assume the following for y^*:

$$\text{(3.1)} \qquad \text{For any } j \in U\colon \begin{cases} f_j(a) \geqq 0 & \text{for all } a \text{ with } y_a^* > 0, \quad \text{or} \\ f_j(a) \leqq 0 & \text{for all } a \text{ with } y_a^* > 0. \end{cases}$$

Next, we may assume that y^* also satisfies

$$\text{(3.2)} \qquad y^* f_j > 0 \Rightarrow c_j > 0,$$

for, suppose $y_a^* > 0$, $f_j(a)=1$ and $c_j \leqq 0$. Let $f_i(a)$ be a -1-component of $f(a)$ immediately preceding or following $f_j(a)$ in the ordered $sf(a)$ (there is such a component if y^*, u^* is optimal). Let $f(b)$ be the alternating vector with components $f_i(b)=$ $=f_j(b)=0$, $f_l(b)=f_l(a)$ for $l \neq i, j$. Change y^*, u^* to y', u^* defined by $y'_a=0$, $y'_b=y_b^*+y_a^*$, $y'_d=y_d^*$ for $d \neq a, b$. y', u^* is also integer, optimal and satisfying (3.1). Applying such changes as long as necessary yields a solution also fulfilling (3.2).

Finally, we may assume that u^* satisfies

$$\text{(3.3)} \qquad u_j^* > 0 \Rightarrow c_j > 0$$

for, if $c_j \leqq 0$ and $u_j^* > 0$, then $y^* f_j < 0$, and there is an a with $y_a^* > 0$ and $f_j(a) = -1$. Let $f_i(a)$ be the $+1$-component of $f(a)$ immediately following $f_j(a)$ in $sf(a)$ and $f(b)$ the alternating vector defined as in the preceding paragraph. y', u' given by $y'_a=y_a^*-\varepsilon$, $y'_b=y_b^*+\varepsilon$, $y'_d=y_d^*$ for $d \neq a, b$, $u'_j=u_j^*-\varepsilon$, $u'_i=u_i^*+\varepsilon$, $u'_l=u_l^*$ for $l \neq i, j$, where $\varepsilon=\min\{y_a^*, u_j^*\}$, is also integer, optimal and satisfying (3.1) and (3.2). Applying such changes when necessary yields a solution also fulfilling (3.3).

Let now $c \in \mathbf{Z}^U$ be given by $c_j=+1$ for $j \in U^+$ and $c_j=-1$ for $j \in U^-$. The duality theorem satisfied with an integer primal solution and an integer dual solution fulfilling (3.1), (3.2) and (3.3) proves the theorem. ∎

3.7. The so called *transition* (or *t*-) *phenomenon*, established for lattice polyhedra ([7], theorem 3.1), also holds for switching paths polyhedra. For instance, for those of type P_2, we have

Theorem 3.4. *Given* P_2 *of theorem* 2.8 *and* $c \in \mathbf{Z}^U$, *the two* lp's

$$(3.4)_{r'} \quad \textit{minimize} \quad r'y + du - ev \quad \textit{s.t.} \quad yf_j + u_j - v_j = c_j, \quad j \in U, y, u, v \geqq 0$$

for $r' = r$ *and* $r' = r+1$, *have a common integer optimal solution whenever they have an optimal solution.*

Proof. Suppose $(3.4)_r$ and $(3.4)_{r+1}$ have an optimum, and let ω_0, respectively ω_1 be their optimal objective value. Note that ω_0 and ω_1 are integer.

Add an element j' to U and an additional column $f_{j'}$ to F with $f_{j'}(a) = -1$ for all $a \in A$, obtaining the augmented matrix F'. Let j' precede any element $j \in U$ in $sf(a)$ and retain the given order $\underset{a}{<}$ in $sf(a)$. Then it is easy to verify that the rows of F' form again a switching family and that

$$(3.5) \qquad \{(x, z)\colon x \in \mathbf{R}^U,\ z \in \mathbf{R},\ e \leqq x \leqq d,\ Fx - z1 \leqq r\}$$

is of type P_2. The proof is now similar to the one in [7] and is only sketched here for this reason. Consider the lp

$$(3.6) \qquad \text{maximize } cx + (\omega_0 - \omega_1)z \quad \text{s.t.} \quad Fx - z1 \leqq r, \quad e \leqq x \leqq d,$$

and its dual

(3.7)

$$\text{minimize } ry + du - ev \quad \text{s.t.} \quad yf_j + u_j - v_j = c_j, \quad j \in U, \quad y1 = \omega_1 - \omega_0, \quad y, u, v \geqq 0.$$

One shows that (3.6) and (3.7) have an optimum of value ω, and, using the fact that (3.6) has an integer optimal solution, that $\omega = \omega_0$. Since the system of (3.5) is tdi and $\omega_1 - \omega_0$ is integer, (3.7) has an integer optimal solution y^*, u^*, v^* with value ω_0, hence optimal for $(3.4)_r$, and, since $y1 = \omega_1 - \omega_0$, also optimal for $(3.4)_{r+1}$. ∎

Applying Theorem 3.4 to the second polyhedron of 3.3 yields the property established by Greene and Kleitman [5] on the existence of a partition of chains which is in their terminology both k- and $k+1$ *saturated* (see also [10]).

For the polyhedron of 3.5, one easily derives from Theorem 3.4 the following.

Corollary 3.5. *Given a partition* U^+, U^- *of the poset* U *and letting* β_k, β_{k+1} *be the maximum surplus w.r.t.* U^+, U^- *of a set partitionable into* k, *respectively* $k+1$ *convex sets, there exists* $S^* \subseteqq U^+$ *and a family* $C_1, \ldots, C_{\alpha(U-S^*)}$ *of alternating chains w.r.t.* $U^+ - S^*$, U^-, *covering* $U^+ - S^*$ *and packed in* U^-, *such that* $\beta_k = |S^*| + k\alpha(U - S^*)$ *and* $\beta_{k+1} = |S^*| + (k+1)\alpha(U - S^*)$. (If the family is empty, set $\alpha(U - S^*) = 0$). ∎

4. Conformity and modularity

For any two vectors $x, y \in \mathbf{R}^U$, x will be said to *conform to* y if $s^+x \subseteqq s^+y$ and $s^-x \subseteqq s^-y$. A crucial lemma in the proof of Theorem 2.8 is the following.

Lemma 4.1. *Let $c \in \mathbf{R}^U$ be integer and $\mathfrak{s} = \{f(a)\colon a \in A\}$ be a modular switching family such that all $f(a)$'s conform to c. Then the system of (in) equalities*

$$\begin{aligned} &\sum_{a\in A} y_a f_j(a) = c_j, \quad j \in U_=, \\ (4.1) \qquad &\sum_{a\in A} y_a f_j(a) \leqq c_j, \quad j \in U_\leqq, \\ &\sum_{a\in A} y_a f_j(a) \geqq c_j, \quad j \in U_\geqq, \\ &y_a \geqq 0, \quad a \in A, \end{aligned}$$

where $U_= \cup U_\leqq \cup U_\geqq$ is an arbitrary partition of U, has an integer solution $\hat{y} \in \mathbf{R}^A$ whenever it has a solution.

Proof. As all $f(a)$'s conform to c, each column f_j is either non-negative or negative. Note that after multiplying the j-th (in) equality by -1 and redefining the partition of U, the hypotheses of the lemma are still fulfilled. We may therefore assume in the following $f_j(a) \geqq 0$ for all $j \in U$ and $a \in A$.

Construct the digraph $G = (V, E)$, where $V = \bigcup_{a\in A} sf(a)$ (hence we may assume $V = U$), and where $(i, j) \in E$ if and only if for some $a \in A$, $f_i(a) = f_j(a) = 1$, $i \underset{a}{<} j$ and there is no k with $i \underset{a}{<} k \underset{a}{<} j$ (and $f_k(a) = 1$): in this sense, j "covers" i in $sf(a)$ and we shall write for this $i \underset{a}{\hat{<}} j$.

i) We show that G is acyclic.

First, not both (i, j) and (j, i) are in E, that is, for any $i \neq j$:

$$(4.2) \qquad \left.\begin{aligned} f_i(a) = f_j(a) = 1 \\ f_i(b) = f_j(b) = 1 \end{aligned}\right\} \Rightarrow i \underset{a}{<} j \ \text{ and } \ i \underset{b}{<} j \ \text{ or } \ i \underset{a}{>} j \ \text{ and } \ i \underset{b}{>} j$$

for, suppose $f_i(d) = f_j(d) = 1$, $d = a, b$, and $i \underset{a}{<} j$ and $i \underset{b}{>} j$. Then $bja \in A$, $f_i(bja) = 1$ by modularity and by (2.1)$'$ $i \in [bja]$, a contradiction.

Next, we prove

$$(4.3) \qquad \left.\begin{aligned} f_i(a) = f_j(a) = 1 \ \text{ and } \ i \underset{a}{<} j \\ f_j(b) = f_k(b) = 1 \ \text{ and } \ j \underset{b}{<} k \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} &f_i(ajb) = f_j(ajb) = f_k(ajb) = 1 \\ &\text{and} \quad i \underset{ajb}{<} j \underset{ajb}{<} k. \end{aligned}\right.$$

Clearly, $ajb \in A$ and by modularity $f_j(ajb) = 1$. Suppose $f_i(ajb) = 0$. By modularity, $f_i(b) = 0$ and $f_i(bja) = 1$, contradicting (2.2)$'$. Similarly, if $f_k(ajb) = 0$, $f_k(a) = 0$ and $f_k(bja) = 1$, contradicting (2.3)$'$. $i \underset{ajb}{<} j \underset{ajb}{<} k$ follows from (4.2). By (4.2) and (4.3), G is acyclic.

ii) Let B and $D \subseteqq V$ be defined by

$$B = \left\{i \in U\colon i \text{ minimal in } sf(a) \text{ with respect to } \underset{a}{<} \text{ for some } a \in A\right\},$$

$$D = \left\{i \in U\colon i \text{ maximal in } sf(a) \text{ with respect to } \underset{a}{<} \text{ for some } a \in A\right\}.$$

We show that for any path P in G from a node $i_0 \in B$ to a node $i_m \in D$ with ordered node set $V(P)=(i_0, i_1, \dots, i_m)$, there exists a $d \in A$ such that $V(P)=sf(d)$ and $i_{r-1} \underset{d}{\hat{<}} i_r$ for $r=1, \dots, m$.

By definition, there is $a \in A$ such that i_0 is minimal in $sf(a)$. Assume $f_{i_s}(a)=1$ for $s=0, \dots, r$ and $i_{s-1} \underset{a}{\hat{<}} i_s$ for $s=1, \dots, r$, and consider the subpath $(i_0, \dots$ $\dots, i_r, i_{r+1})$. Since $(i_r, i_{r+1}) \in E$, there exists $b \in A$ such that $i_r \underset{b}{\hat{<}} i_{r+1}$. We may assume $b \neq a$. For ease of notation and up to the end of the proof, denote ai_rb by e. Clearly, $e \in A$ and from (4.3) follows $f_{i_s}(e)=1$ for $s=0, \dots, r+1$, and from (4.2) $i_{s-1} \underset{e}{<} i_s$ for $s=1, \dots, r+1$. Further, i_0 is the minimal element of $sf(e)$: suppose there is i with $f_i(e)=1$ and $i \underset{e}{<} i_0$. Then $f_i(a)=0$, otherwise by (4.2) $i \underset{a}{<} i_0$, a contradiction. But then $f_i(b)=1$ by modularity, and $i \underset{b}{<} i_r$ by (4.2), contradicting (2.2).

Also, $i_{s-1} \underset{e}{\hat{<}} i_s$ for $s=1, \dots, r+1$: suppose there is i with $f_i(e)=1$ and $i_{s-1} \underset{e}{<} i \underset{e}{<} i_s$ for some $s \in \{1, \dots, r\}$. Again $f_i(a)=0$, other $i_{s-1} \underset{a}{<} i \underset{a}{<} i_s$. Then $f_i(b)=1$ and $i \underset{b}{<} i_r$, contradicting (2.2). If $f_i(e)=1$ and $i_r \underset{e}{<} i \underset{e}{<} i_{r+1}$, $f_i(b)=0$, otherwise $i_r \underset{b}{<} i \underset{b}{<} i_{r+1}$, hence $f_i(a)=1$ and $i_r \underset{a}{<} i$, contradicting (2.3).

By induction on r, there is $g \in A$ such that i_0 is minimal in $sf(g)$, $\{i_0, \dots, i_m\} \subseteq$ $\subseteq sf(g)$ and $i_{s-1} \underset{g}{\hat{<}} i_s$, $s=1, \dots, m$. Since $i_m \in D$, there is $h \in A$ such that i_m is the maximal element of $sf(h)$. $gi_mh \in A$ and with the same arguments as above, one shows that gi_mh is the desired d.

iii) Construct the auxiliary digraph G' obtained from G by introducing a source s, a sink t and additional edges (s, j) for all $j \in B$ and (j, t) for all $j \in D$. G' is acyclic. Make G' to a node constraint network by setting the following lower and upper bounds on the nodes of G': $l_s=l_t=0$, $u_s=u_t=\infty$, $l_j=u_j=c_j$ for $j \in U_=$, $l_j=0$ and $u_j=c_j$ for $j \in U_{\leqq}$, $l_j=c_j$ and $u_j=\infty$ for $j \in U_{\geqq}$. To a solution $y \in \mathbf{R}^A$ of the system (4.1) corresponds a feasible $s-t$-flow. Since c is integer, there exists an *integer* feasible $s-t$-flow in G'. Any integral path-decomposition of this integer flow yields then an integer solution $\hat{y}$ to (4.1). ▮

A general lemma allows to restrict ourselves in the coming proof of Theorem 2.8 to the case where there are no lower or upper bounds on the variables. It characterizes the property of box total dual integrality. Let $Q \in \mathbf{R}^{m \times n}$ be a $m \times n$-matrix and $q \in \mathbf{R}^m$. Edmonds and Giles [3] call a system $Qx \leqq q$ *box totally dual integral* (box tdi) if the system $Qx \leqq q$, $e \leqq x \leqq d$ is tdi for any vectors $d, e \in \{\mathbf{R} \cup \pm\infty\}^n$.

Lemma 4.2. *For a system* $Qx \leqq q$, *(i) and (ii) are equivalent:*

(i) *For any* $J \subseteq \{1, \dots, n\}$ *and* $b_j \in \mathbf{R}$, $j \in J$, $Qx \leqq q$, $x_j=b_j$, $j \in J$ *is tdi.*
(ii) $Qx \leqq q$ *is box tdi.*

Proof of (i)⇒(ii). ((ii)⇒(i) is obvious).

a) We show first (i)⇒(ii) for the case where the box in (ii) consists of a single inequality, i.e. assuming (i), we show that $Qx \leqq q$, $x_1 \leqq d_1$ is tdi. (A proof of $Qx \leqq q$,

$e_1 \leqq x_1$ tdi is similar). Therefore let $c \in \mathbf{Z}^n$ and assume that the (dual) lp

$$\text{(4.4)} \qquad \begin{aligned} &\text{minimize } qy + d_1 u, \quad y \in \mathbf{R}^m, \quad u \in \mathbf{R}, \\ &\text{s.t.} \quad yq^1 + u = c_1, \quad yq^j = c_j, \quad j = 2, \ldots, n \\ &\qquad y \geqq 0, \quad u \geqq 0, \end{aligned}$$

has an optimal objective value ω (q^j denotes the j-th column of Q). If (4.4) has any optimal solution y^0, u^0 with $u^0 = 0$, it has an integer optimal solution $y^*, 0$, since $Qx \leqq q$ is tdi. So assume the contrary and let y^0, u^0 be some optimal solution. By complementary slackness $x_1^* = d_1$ for any optimal solution x^* of the corresponding primal lp. Therefore $\omega = \max\{cx\colon Qx \leqq q, \; x_1 = d_1\} = \min\{qy + d_1 u\colon yq^1 + u = c_1, yq^j = c_j, j = 2, \ldots, n, y \geqq 0\}$. The latter minimum is achieved by some integer solution $\hat{y}, \hat{u}$, since (i) holds for $Qx \leqq q$. Now, $\hat{u}$ cannot be negative, otherwise $\tilde{y}, 0$ defined by $\tilde{y} = \alpha\hat{y} + (1-\alpha)y^0$ and $\alpha = u^0/(u^0 - \hat{u})$ is an optimal solution to (4.4), contradicting the above assumption. Therefore $\hat{u} \geqq 0$ and $\hat{y}, \hat{u}$ is also an integer optimal solution to (4.4).

b) Incorporate $x_1 \leqq d_1$ (or $e_1 \leqq x_1$) into $Qx \leqq q$, obtaining $Q'x \leqq q'$. Then (i) also holds for $Q'x \leqq q'$. To see this, consider for some $J \subseteqq \{1, \ldots, n\}$ and $b_j \in \mathbf{R}$, $j \in J$, the system

$$\text{(4.5)} \qquad \begin{aligned} &Qx \leqq q, \quad x_j = b_j, \quad j \in J, \\ &x_1 \leqq d_1. \end{aligned}$$

(i) for $Qx \leqq q$ obviously implies (i) for the system $Qx \leqq q$, $x_j = b_j$, $j \in J$; applying the result of a) to this latter system proves that (4.5) is tdi. Successive incorporations of all box constraints and use of a) establish the lemma. ∎

Remark 4.3. An equivalent form of (i) of Lemma 4.2 is:

(i′) *For any* $b \in \mathbf{R}^n$ *such that* $Qb \leqq q$ *and* $J \subseteqq \{1, \ldots, n\}$

$$\text{(4.6)} \qquad \sum_{j \notin J} q^j x_j \leqq q - \sum_{j \in J} q^j b_j$$

is tdi.

This observation makes it easy to deduce the box tdi-property of certain systems from their tdi-property. More specifically, if a class $\mathfrak{C}$ of tdi-systems is such that for any member $Qx \leqq q$ of $\mathfrak{C}$, the system (4.6) is also a member of $\mathfrak{C}$, then $\mathfrak{C}$ is a class of box-tdi systems. This argument applies to several known classes of systems (for instance of [3], [8]) and also to the two classes describing the switching paths polyhedra P_1 and P_2 of theorem 2.8: for instance in $x \geqq 0$, $Fx \geqq r$, the deletion of certain columns f_j from F and the subtraction from r of a non negative combination of these columns yields a system $x' \geqq 0$, $Fx' \geqq r'$ of the same type.

5. Proof of the theorem

We shall prove that the system describing P_1 with $e=0$ and $d=+\infty$ is tdi. By the preceding remark, the system will then be tdi for any e and d. Let $c\in\mathbf{R}^U$ be integer and consider the lp

$$\text{(5.1)}\qquad \begin{aligned} &\text{minimize } cx, \quad x\in\mathbf{R}^U,\\ &\text{s.t.} \quad \sum_{j\in U} f_j(a)x_j \geqq r(a), \quad a\in A,\\ &\qquad x \geqq 0, \end{aligned}$$

and its dual

$$\text{(5.2)}\qquad \begin{aligned} &\text{maximize } ry, \quad y\in\mathbf{R}^A,\\ &\text{s.t.} \quad \sum_{a\in A} y_a f_j(a) \leqq c_j, \quad j\in U,\\ &\qquad y \geqq 0. \end{aligned}$$

Assume (5.1) and (5.2) have a finite optimum and let x^* be an optimal solution to (5.1). Define the sets $A_= = \{a\in A\colon \sum f_j(a)x_j^* = r(a)\}$ and $U_= = \{j\in U\colon x_j^* > 0\}$. By standard results of linear programming, a solution to (5.2) is optimal iff it satisfies the system:

$$\text{(5.3)}\qquad \begin{aligned} &\sum_{a\in A_=} y_a f_j(a) = c_j, \quad j\in U_=,\\ &\sum_{a\in A_=} y_a f_j(a) \leqq c_j, \quad j\in U-U_=,\\ &y_a \geqq 0, \quad a\in A_=,\\ &y_a = 0, \quad a\in A-A_=. \end{aligned}$$

We will find a subfamily $A_*\subseteqq A_=$ and $c^*\in\mathbf{R}^U$ integer such that

$$\text{(5.4)}\qquad \begin{aligned} &\sum_{a\in A_*} y_a f_j(a) = c_j^* = c_j, \quad j\in U_=,\\ &\sum_{a\in A_*} y_a f_j(a) \leqq c_j^* \leqq c_j, \quad j\in U-U_=,\\ &y_a \geqq 0, \quad a\in A_*, \end{aligned}$$

admits a feasible solution y^*, and such that $\{f(a)\colon a\in A_*\}$ is a switching family whose $f(a)$'s conform to c^*, and whose vectors $f_j\in\mathbf{R}^{A^*}$, $j\in U$, are modular on A_*. Then by Lemma 4.1, (5.4) and hence (5.3) have an integer feasible, and hence (5.2) an integer optimal solution. Let y^0 be an optimal solution to (5.2) and define the sets

$$Y_0 = \{y\in\mathbf{R}^A\colon y \text{ feasible in (5.3) and } 1y = 1y^0\},$$

$$Y_1 = \Big\{y\in Y_0\colon y \text{ minimizes } z(y) \equiv \sum_{j\notin U_=} \sum_{a\in A_=} f_j(a)y_a\Big\},$$

and for any $j\in U$ and $y\in\mathbf{R}^A$, the integer number $\gamma_j(y) = |\{(a,b)\colon y_a \text{ and } y_b>0, \text{ and } f_j(y) = -f_j(b)\neq 0\}|$. Order U from 1 to n, define $Y_2=\{y\in Y_1\colon y$ minimizes lexicographically $\gamma_1(y), \gamma_2(y), \ldots\}$, and choose $y^*\in Y_2$ with a maximum number of non zero components. For any scalar α, denote by $\lceil\alpha\rceil$ the largest integer less than or equal to α. We claim that the desired A_* and c^* can be chosen as

$$A_* = \{a \in A_=: y_a^* > 0\}, \quad \text{and} \tag{5.5}$$

$$c_j^* = \lceil \sum f_j(a) y_a^* \rceil, \quad j \in U. \tag{5.6}$$

First, if $y^1 \in Y_1$, then for any a, b such that $y_a^1, y_b^1 > 0$ and $j \in sf(a) \cap sf(b)$, one has ajb and $bja \in A_=$ and

$$f_i(a) + f_i(b) = f_i(ajb) + f_i(bja) \quad \text{for all} \quad i \in U. \tag{5.7}$$

To see this, derive y' from y^1 be the following "ε-changes":

$$y'_d = \begin{cases} y_d^1 - \varepsilon & \text{for} \quad d = a, b, \\ y_d^1 + \varepsilon & \text{for} \quad d = ajb, bja \\ y_d^1 & \text{for all other} \quad d \in A, \end{cases}$$

where $0 < \varepsilon \leqq \min\{y_a^1, y_b^1\}$. Then y' is optimal for (5.2), as is easily seen using submodularity of the f_j's and supermodularity of r. Also $1y' = 1y^1$, therefore $y' \in Y_0$. As a consequence, ajb and $bja \in A_=$ and (5.7) holds for $i \in U_=$. Also, if for some $i \in U - U_=$, $f_i(a) + f_i(b) > f_i(ajb) + f_i(bja)$, then $z(y') < z(y^1)$, contradicting $y^1 \in Y_1$. Observe that, if $y^1 \in Y_1$, then also $y' \in Y_1$.

Next, if $y^2 \in Y_2$, then $\gamma_j(y^2) = 0$ for all $j \in U$, for suppose the contrary and let j be the smallest index for which $\gamma_j(y^2) > 0$. We may assume $0 < y_a^2 \leqq y_b^2$ and $f_j(a) = -f_j(b) \neq 0$ for some $a, b \in A_=$. Derive y'' from y^2 by ε-changes, where now $\varepsilon = y_a^2$. By (2.2), (2.3), (2.2)$'$ and (2.3)$'$, $f_j(ajb) = f_j(bja) = 0$, hence $\gamma_j(y'') < \gamma_j(y^2)$. Also, $\gamma_i(y^2) = 0$ implies $\gamma_i(y'') = 0$, for suppose $\gamma_i(y'') > 0$. Necessarily, $f_i(ajb) = -f_i(d) \neq 0$ or $f_i(bja) = -f_i(d) \neq 0$ for some d with $y''_d > 0$. Assume the first case, the second being similar. If $d = bja$, by (2.4) and modularity, $f_i(a) = -f_i(b) \neq 0$, contradicting $\gamma_i(y^2) = 0$. If $d = a$ or b, by modularity, again $f_i(a) = -f_i(b) \neq 0$. If $d \notin \{a, b, bja\}$, by (2.4) and modularity, $f_i(a)$ or $f_i(b) = -f_i(d)$, again contradicting $\gamma_i(y^2) = 0$. As a result, $y'' \in Y_1$ (be the observation ending the precedent paragraph) and y'' is lexicographically smaller than y^2, a contradiction.

Finally, with successive ε-changes, one derives y^* from y^2 such that $y^* \in Y_1$, $\gamma_j(y^*) = 0$ for all $j \in U$, and, A_* being given by (5.5), $a, b \in A_*$ implies ajb and $bja \in A_*$. As all $\gamma_j(y^*) = 0$, $f(a)$, $a \in A_*$, obviously conform to c^* of (5.6) and A_*, c^* and y^* are as desired for the system (5.4).

The proof of b) of Theorem 2.8 is similar. ▌

Remark 5.1. In the work of Cameron [1], the class of so called *coflow polyhedra* is introduced and used in a number of applications. It would be interesting to find a common frame for both coflow and switching paths polyhedra.

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